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Integrable multi-boson systems and orthogonal polynomials

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Abstract

The strict relation between a certain class of multi-boson Hamiltonian systems and the corresponding class of orthogonal polynomials is established. The correspondence is effectively used to integrate the systems. As an explicit example we integrate the class of multi-boson systems corresponding to q-Hahn class polynomials.

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Introduction

The results of this paper provide an effective tool to integrate a broad class of quantum physical systems. Such systems play a prominent role in quantum many-body physics, nuclear physics as well as quantum optics. More information about these models and their physical content can be found in many papers and monographs on this subject and among others one may consult [A-I, J1, J2, Kar1, Kar2, M-G, Odz].

This paper is devoted to a detailed study of the mathematical structures underlying multi-boson Hamiltonian systems, whose dynamics are generated by the operators of the general form given in (1.1).

In the first section it is shown that the reduction procedure applied to a multi-boson Hamiltonian system leads to some interesting operator algebras $\mathcal{A}_{\mathcal{R}}$ (parametrized by a structural function $\mathcal{R}$), which generalize in a natural way Heisenberg algebra as well as $sl_q(2)$ and $SU_q(2)$ quantum algebras. It was demonstrated [Odz] that $\mathcal{A}_{\mathcal{R}}$ algebras are on one hand strictly related to the integration of quantum systems but on the other hand, their description in terms of coherent states leads to the connection of these algebras with the theory of basic hypergeomeric series and orthogonal polynomials.

In section 2 we establish the explicit relation between the spectral realization of $\mathcal{A}_{\mathcal{R}}$ algebra in $L^2(\mathbb{R}, \mathrm{d}\sigma)$ space and representation in holomorphic function space.

According to spectral theorem, the problem of the integration of dynamical systems is equivalent to the construction of the spectral measure $\mathrm{d}\sigma$ for the corresponding Hamiltonian. By making certain assumptions about its form (see section 3) the problem of this construction can be explicitly solved within the framework of q-Hahn's polynomials theory. The measure is then uniquely determined by solving the q-difference Pearson equation.

For the sake of completeness some elementary facts of q-analysis are presented in appendix A. Appendix B contains the proofs of some fundamental properties of the q-Hahn class polynomials.

1. Multi-boson systems

This section is devoted to the detailed analysis of the symmetry properties of $(N+1)$-boson systems. The dynamics of these systems is assumed to be governed by the Hamiltonian operator of the form

$$H = h_0\left(a_0^*a_0,\dots,a_N^*a_N\right) + g_0\left(a_0^*a_0,\dots,a_N^*a_N\right)a_0^{k_0}\dots a_N^{k_N} \\ +a_0^{-k_0}\dots a_N^{-k_N}\overline{g}_0\left(a_0^*a_0,\dots,a_N^*a_N\right) \tag{1.1}$$

where $a_0,\dots,a_N$ and $a_0^*,\dots,a_N^*$ are bosonic annihilation and creation operators with the standard Heisenberg commutation relations

$$\left[a_i,a_j^*\right]=\delta_{ij} \qquad \left[a_i,a_j\right]=0 \qquad \left[a_i^*,a_j^*\right]=0. \tag{1.2}$$

The following notational convention is assumed in (1.1):

$$a_i^{k_i} = \begin{cases} a_i^{k_i} & \text{for } k_i>0 \\ 1 & \text{for } k_i=0 \\ (a_i^*)^{-k_i} & \text{for } k_i<0. \end{cases} \tag{1.3}$$

The monomial

$$a_0^{k_0}\dots a_N^{k_N} \qquad k_0,\dots,k_N\in\mathbb{Z} \tag{1.4}$$

can be thought of as an operator which describes the subsequent creation and annihilation of clusters of the bosonic modes. The operator

$$g_0\left(a_0^*a_0,\dots,a_N^*a_N\right) \tag{1.5}$$

is a kind of generalization of the coupling constant. The coupling constant is replaced in our case by a function depending on the commuting occupation number operators of the bosonic modes. The operator

$$h_0\left(a_0^*a_0,\dots,a_N^*a_N\right) \tag{1.6}$$

can always be chosen as a free Hamiltonian being a weighted sum of the occupation number operators of the elementary modes $a_0^*a_0,\dots,a_N^*a_N$

$$h_0^{\text{free}} = \omega_0a_0^*a_0+\dots+\omega_Na_N^*a_N. \tag{1.7}$$

This paper will by no means be restricted to this free case, however. The Hamiltonian under consideration (1.1) is an elementary ingredient of the most general Hamiltonian operator

$$H = \sum_{k_0,\dots,k_N\in\mathbb{Z}} g_{k_0\dots k_N}\left(a_0^*a_0,\dots,a_N^*a_N\right)a_0^{k_0}\dots a_N^{k_N} \tag{1.8}$$

with the functions $g_{k_0\dots k_N}$ being connected by the following conjugation rule:

$$\left[g_{k_0\dots k_N}\left(a_0^*a_0,\dots,a_N^*a_N\right)\right]^* = g_{-k_0\dots -k_N}\left(a_0^*a_0-k_0,\dots,a_N^*a_N-k_N\right). \tag{1.9}$$

The class of model Hamiltonians (1.1) corresponds to many important quantum physical systems. Their dynamics is generated by specific operators of the form (1.1) [J1, J2, Kar1, Kar2, A-I]. For this reason the analysis of the system (1.1) seems to be important and may shed new light on the unsolved problems of quantum physics.

In order to analyse the quantum system described by the Hamiltonian (1.1), it is convenient to introduce the operators

$$A := g_0\left(a_0^* a_0, \ldots, a_N^* a_N\right) a_0^{k_0} \ldots a_N^{k_N} \tag{1.10}$$

and

$$A_i = A_i^* := \sum_{j=0}^{N} \alpha_{ij} a_j^* a_j \tag{1.11}$$

where $i = 0, 1, 2, \ldots, N$. One assumes that the real $(N+1) \times (N+1)$-matrix $\alpha = (\alpha_{ij})$ satisfies the conditions

$$\det \alpha \neq 0 \tag{1.12}$$

$$\sum_{j=0}^{N} \alpha_{ij} k_j = \delta_{i0}. \tag{1.13}$$

The operators $A_0, A_1, \ldots, A_N$, A and A^* satisfy the commutation relations

$$[A_0, A] = -A \qquad [A_0, A^*] = A^* \tag{1.14}$$

$$[A, A_i] = 0 = [A^*, A_i] \tag{1.15}$$

for $i = 1, \ldots N$, and

$$[A_i, A_j] = 0 \tag{1.16}$$

for $i, j = 0, 1, \ldots, N$.

One also has

$$A^* A = \left|g_0\left(a_0^* a_0 - k_0, \ldots, a_N^* a_N - k_N\right)\right|^2 \mathcal{P}_{k_0}\left(a_0^* a_0 - k_0\right) \ldots \mathcal{P}_{k_N}\left(a_N^* a_N - k_N\right) \tag{1.17}$$

$$A A^* = \left|g_0\left(a_0^* a_0, \ldots, a_N^* a_N\right)\right|^2 \mathcal{P}_{k_0}\left(a_0^* a_0\right) \ldots \mathcal{P}_{k_N}\left(a_N^* a_N\right) \tag{1.18}$$

where $\mathcal{P}_{k_0}\left(a_0^* a_0\right), \ldots, \mathcal{P}_{k_N}\left(a_N^* a_N\right)$ are the polynomials given by

$$\mathcal{P}_k\left(a^* a\right) := a^k a^{-k} = \begin{cases} a^k \left(a^*\right)^k = \left(a^* a + 1\right) \ldots \left(a^* a + k\right) & \text{for} \quad k > 0 \\ 1 & \text{for} \quad k = 0 \\ \left(a^*\right)^{-k} a^{-k} = a^* a \left(a^* a - 1\right) \ldots \left(a^* a + k + 1\right) & \text{for} \quad k < 0. \end{cases} \tag{1.19}$$

The operators $A^* A$ and $A A^*$ are diagonals in the standard Fock basis

$$|n_0, n_1, \ldots, n_N\rangle = \frac{1}{\sqrt{n_0! \ldots n_N!}} \left(a_0^*\right)^{n_0} \ldots \left(a_N^*\right)^{n_N} |0\rangle \tag{1.20}$$

where $(n_0, n_1, \ldots, n_N) \in \mathbb{Z}_+^{N+1} := (\mathbb{Z}_+ \cup \{0\}) \times \cdots \times (\mathbb{Z}_+ \cup \{0\})$ ($N + 1$ times).

Let us note that the operators $A_0, \ldots, A_N$ are unbounded. Whether A and A^* are bounded or not depends on the choice of the structural function g_0. All the operators (1.10), (1.11) are defined on the common domain D spanned by the finite linear combinations

$$|v\rangle = \sum_{(i_0, i_1, \ldots, i_N) \in F} c_{i_0, i_1, \ldots, i_N} \left|n_{i_0}, n_{i_1}, \ldots, n_{i_N}\right\rangle \tag{1.21}$$

of the Fock basis elements, where F is a finite set of multi-indices.

Identifying the one-dimensional spaces $\mathbb{C}\,|n_0, n_1, \ldots, n_N\rangle$ with the elements of $\mathbb{Z}_+^{N+1}$ we obtain the action of A and A^* (and their natural powers) on $\mathbb{Z}_+^{N+1}$. It is easy to see that the orbits of these actions are located on one-dimensional lines which are parallel to the vector $(k_0, k_1, \ldots, k_N) \in \mathbb{Z}^{N+1}$.

If the function g_0 in (1.1) is regular and nonvanishing in all points of $\mathbb{Z}_+^{N+1}$, we can make some simple but useful observations. The first one is that if $\frac{k_i}{k_j} < 0$ for some $i, j \in \{0, 1, \ldots, N\}$ then for any element $|n_0, n_1, \ldots, n_N\rangle$ of the Fock basis there exists $M \in \mathbb{N}$ such that

$$A^M |n_0, n_1, \ldots, n_N\rangle = 0 \qquad \text{and} \qquad (A^*)^M |n_0, n_1, \ldots, n_N\rangle = 0. \quad (1.22)$$

This means that the orbits of A and A^* in $\mathbb{Z}_+^{N+1}$ are finite. For the opposite case the orbits contain infinitely many points.

The second observation is that $A|n_0, n_1, \ldots, n_N\rangle = 0$ if and only if there exists $i \in \{0, 1, \ldots, N\}$ such that $k_i > 0$ and $n_i \in \{0, 1, \ldots, k_i - 1\}$. Hence each orbit of A and A^* in $\mathbb{Z}_+^{N+1}$ has exactly one vacuum (the point annihilated by A). It is then natural to introduce the parametrization of the Fock basis in accordance with the above orbit decomposition of $\mathbb{Z}_+^{N+1}$.

Replacing the occupation number operators $a_0^* a_0, \ldots, a_N^* a_N$ by the operators $A_0, A_1, \ldots, A_N$ in (1.17) and (1.18) one obtains

$$A^* A = \mathcal{G}(A_0 - 1, A_1, \ldots, A_N) \quad (1.23)$$

$$A A^* = \mathcal{G}(A_0, A_1, \ldots, A_N) \quad (1.24)$$

with the function $\mathcal{G}$ uniquely determined by g_0, the polynomials $\mathcal{P}_{k_0}, \ldots, \mathcal{P}_{k_N}$ and the linear map (1.11).

The Hamiltonian (1.1) can be re-expressed in terms of (1.10) and (1.11) in the following form:

$$H = H_0(A_0, A_1, \ldots, A_N) + A + A^*. \quad (1.25)$$

It is clear that it admits N commuting integrals of motion $A_1, \ldots, A_N$: $[A_i, H] = 0$ for $i = 1, \ldots, N$ and they also commute with the operator A_0. This maximal system of commuting observables is diagonalized in the Fock basis and the eigenvalues of $A_0, A_1, \ldots, A_N$ on $|n_0, n_1, \ldots, n_N\rangle$ are given by

$$\lambda_i = \sum_{j=0}^{N} \alpha_{ij} n_j \qquad i = 0, 1, \ldots, N. \quad (1.26)$$

The eigenvalues $(\lambda_0, \lambda_1, \ldots, \lambda_N)$ form a discrete convex cone $\Lambda_+ \subset \mathbb{R}^{N+1}$, which is spanned by the columns of the matrix (α_{ij}) with entries from $(\mathbb{Z}_+ \cup \{0\})$: $\Lambda_+ = \alpha\left(\mathbb{Z}_+^{N+1}\right)$. The sequences $(\lambda_0, \lambda_1, \ldots, \lambda_N) \in \Lambda_+$ will be used as a new parametrization $\{|\lambda_0, \lambda_1, \ldots, \lambda_N\rangle\}$ of the Fock basis elements.

In order to integrate system (1.1) one can reduce it to the eigensubspaces $\mathcal{H}_{\lambda_1 \ldots \lambda_N} \subset \mathcal{H}$ spanned by the eigenvectors $|\lambda_0, \lambda_1, \ldots, \lambda_N\rangle$ with fixed $\lambda_1, \ldots, \lambda_N$. Every such subspace is invariant with respect to $\mathcal{A}_{\text{red}}$ algebra, which is generated by the operators A, A^* and A_0. These operators satisfy relations (1.23), (1.24), (1.14)

$$[A_0, A] = -A \qquad [A_0, A^*] = A^* \quad (1.27)$$

$$A^* A = \mathcal{G}(A_0 - 1, \lambda_1, \ldots, \lambda_N) \quad (1.28)$$

$$A A^* = \mathcal{G}(A_0, \lambda_1, \ldots, \lambda_N). \quad (1.29)$$

Hence one can conclude that the problem of integration of the system (1.1) amounts to integration of the system described by the reduced Hamiltonian

$$H_{\text{red}} = H_0(A_0, \lambda_1, \ldots, \lambda_N) + A + A^* \quad (1.30)$$

being an element of $\mathcal{A}_{\text{red}}$ algebra.

The orthonormal basis of the Hilbert subspace $\mathcal{H}_{\lambda_1 \ldots \lambda_N}$ is formed by the vectors $|\lambda_0, \lambda_1, \ldots, \lambda_N\rangle$ with λ_0 such that $(\lambda_0, \lambda_1, \ldots, \lambda_N) \in \Lambda_+$. From (1.27)–(1.29) it follows that

$$A_0 \,|\lambda_0, \lambda_1, \ldots, \lambda_N\rangle = \lambda_0 \,|\lambda_0, \lambda_1, \ldots, \lambda_N\rangle \tag{1.31}$$

$$A \,|\lambda_0, \lambda_1, \ldots, \lambda_N\rangle = \sqrt{\mathcal{G}\,(\lambda_0 - 1, \lambda_1, \ldots, \lambda_N)}|\lambda_0 - 1, \lambda_1, \ldots, \lambda_N\rangle \tag{1.32}$$

$$A^*|\lambda_0, \lambda_1, \ldots, \lambda_N\rangle = \sqrt{\mathcal{G}\,(\lambda_0, \lambda_1, \ldots, \lambda_N)}|\lambda_0 + 1, \lambda_1, \ldots, \lambda_N\rangle. \tag{1.33}$$

Let us note here that if $(\lambda_0, \lambda_1, \ldots, \lambda_N) \in \Lambda_+$, then either $(\lambda_0 - 1, \lambda_1, \ldots, \lambda_N) \in \Lambda_+$ $((\lambda_0 + 1, \lambda_1, \ldots, \lambda_N) \in \Lambda_+)$ or

$$A \,|\lambda_0, \lambda_1, \ldots, \lambda_N\rangle = 0 \qquad \big(A^* \,|\lambda_0, \lambda_1, \ldots, \lambda_N\rangle = 0\big)\,. \tag{1.34}$$

Due to (1.28), (1.29) conditions (1.34) are equivalent to

$$\mathcal{G}\,(\lambda_0 - 1, \lambda_1, \ldots, \lambda_N) = 0 \qquad (\mathcal{G}\,(\lambda_0, \lambda_1, \ldots, \lambda_N) = 0)\,. \tag{1.35}$$

Since Λ_+ is a discrete convex cone we can easily see that the representation of $\mathcal{A}_{\text{red}}$ in $\mathcal{H}_{\lambda_1\ldots\lambda_N}$ splits into irreducible components. These components are generated out of the vacuum (or antivacuum) states $|\lambda_0, \lambda_1, \ldots, \lambda_N\rangle$. The vacuum states are parametrized by the solutions λ_0 of equations (1.35). In all cases under consideration the operator A_0 is diagonal while the operator A is a weighted unilateral shift operator. One does not exclude the case when the irreducible representations generated by $\big|\lambda_0, \lambda_1, \ldots, \lambda_N\big\rangle$ are of finite dimension.

We can give now the following proposition.

Proposition 1.1. *If the structural function $\mathcal{G}$ is regular then:*

(1) $\dim \mathcal{H}_{\lambda_1\ldots\lambda_N} < \infty$ *if and only if there exists a pair $i, j \in \{0, 1, \ldots, N\}$ such that $\frac{k_i}{k_j} < 0$.*

(2) if $k_0 > 0$ then the equation $A\,|\lambda_0, \lambda_1, \ldots, \lambda_N\rangle = 0$ is solved by

$$\lambda_{0,l} := \frac{l}{k_0} - \frac{1}{k_0}\sum_{j=1}^{N}\beta_{0j}\lambda_j \tag{1.36}$$

where β_{ij} are matrix elements of α^{-1}, $l \in L := \big\{0, \frac{1}{\kappa}k_0, \frac{2}{\kappa}k_0, \ldots, \frac{\kappa-1}{\kappa}k_0\big\}$ and κ is the greatest common divisor of the numbers $(k_0, k_1, \ldots, k_N)$. Moreover $\mathcal{H}_{\lambda_1\ldots\lambda_N}$ splits into the irreducible components

$$\mathcal{H}_{\lambda_1\ldots\lambda_N} = \bigoplus_{l\in L}\mathcal{H}^l_{\lambda_1\ldots\lambda_N} \tag{1.37}$$

where $\mathcal{H}^l_{\lambda_1,\ldots,\lambda_N}$ are generated by $\mathcal{A}_{\text{red}}$ out of the states $\big|\lambda_{0,l}, \lambda_1, \ldots, \lambda_N\big\rangle$.

(3) $\dim \mathcal{H}_{\lambda_1,\ldots,\lambda_N} = \infty$ *if and only if* $\dim \mathcal{H}^l_{\lambda_1,\ldots,\lambda_N} = \infty$ *for all l.*

Proof. This follows immediately from the observations above. The irreducibility of $\mathcal{H}^l_{\lambda_1\ldots\lambda_N}$ is a consequence of the fact that $\big|\lambda_{0,l}, \lambda_1, \ldots, \lambda_N\big\rangle$ is a unique vacuum in this space. □

The construction presented above generalizes that of [Kar1, Kar2], where models with $\mathcal{G}$ as a special polynomial are considered. In these papers perturbation integration methods were used.

As mentioned in the Introduction, our aim is to study an integrable family of Hamiltonians (1.30) and thus, instead of the operator A_0, we will use the Hermitian operator

$$Q := q^{A_0 - \lambda_{0,l}} \tag{1.38}$$

where $0 < q < 1$.

The relations (1.27)–(1.29) then acquire the following form in terms of the operators A, A^* and Q:

$$QA^* = qA^*Q, \quad qQA = AQ \tag{1.39}$$

$$A^*A = \mathcal{R}\,(Q) \tag{1.40}$$

$$AA^* = \mathcal{R}\,(qQ) \tag{1.41}$$

where the structural function $\mathcal{R}$ is given by

$$\mathcal{R}(Q) = \mathcal{G}\left(\frac{\log Q}{\log q} + \lambda_{0,l} - 1, \lambda_1, \ldots, \lambda_N\right). \tag{1.42}$$

The function $\mathcal{R}$ takes positive values in all points $\{q^n\}_{n=1}^{\infty}$ and moreover $\mathcal{R}(1) = 0$. Algebras of this type were analysed in [Odz]. The reduced Hamiltonian (1.30) can be rewritten as

$$H_{\rm red} = \mathcal{D}(Q) + A + A^* \tag{1.43}$$

were the function $\mathcal{D}$ is given as

$$\mathcal{D}(Q) = H_0\left(\frac{\log Q}{\log q} + \lambda_{0,l}, \lambda_1, \ldots, \lambda_N\right) \tag{1.44}$$

by the use of (1.38).

Using (1.42), (1.41), (1.38), (1.29), (1.18) and (1.11) we can reconstruct the coupling function g_0 (see Hamiltonian (1.1)) out of the function $\mathcal{R}$

$$\left|g_0\left(a_0^*a_0, \ldots, a_N^*a_N\right)\right|^2 = \frac{\mathcal{R}\left(q^{1-\lambda_{0,l}+\sum_{j=0}^{N}\alpha_{0j}a_j^*a_j}\right)}{\mathcal{P}_{k_0}\left(a_0^*a_0\right)\ldots\mathcal{P}_{k_N}\left(a_N^*a_N\right)}. \tag{1.45}$$

The reconstruction of h_0 is not possible in this case because of the lack of formulae of the type (1.41) and (1.18).

The analysis below is restricted only to the infinite-dimensional case. The discussion of the finite-dimensional case will be presented in a separate paper.

2. Spectral and holomorphic representations

Let $\mathcal{A}_{\mathcal{R}}$ be the operator algebra generated by the operators A, A^* and Q. In this section we describe two natural and, importantly from a physical point of view, representations of $\mathcal{A}_{\mathcal{R}}$ algebra.

The first representation, which we will call a holomorphic one, is related to the coherent states $|z\rangle$, $|z| < \mathcal{R}(0)$, of the annihilation operator $A \in \mathcal{A}_{\mathcal{R}}$

$$A|z\rangle = z|z\rangle. \tag{2.1}$$

Let us consider the case when the orthonormal basis of the Hilbert space $\mathcal{H}_{\rm red} := \mathcal{H}^l_{\lambda_1\ldots\lambda_N}$

$$|n\rangle := \left|\lambda_{0,l} + n, \lambda_1, \ldots, \lambda_N\right\rangle \qquad n \in \mathbb{N}\cup\{0\} \tag{2.2}$$

is infinite. The vectors $|n\rangle$ are generated by the operator A^* out of the vacuum state $|0\rangle$. The action of the algebra generators on the vectors of this basis is

$$Q|n\rangle = q^n|n\rangle \tag{2.3}$$

$$A|n\rangle = \sqrt{\mathcal{R}(q^n)}\,|n-1\rangle \tag{2.4}$$

$$A^*|n\rangle = \sqrt{\mathcal{R}\left(q^{n+1}\right)}|n+1\rangle. \tag{2.5}$$

The coherent state $|z\rangle$ is thus given by

$$|z\rangle := \sum_{n=0}^{\infty}\frac{z^n}{\sqrt{\mathcal{R}(q)\ldots\mathcal{R}(q^n)}}\,|n\rangle \tag{2.6}$$

where $z \in \mathbb{D} = \{z \in \mathbb{C} : |z| < \mathcal{R}(0)\}$. The states $|z\rangle$, $z \in \mathbb{D}$ form a linearly dense subset in $\mathcal{H}_{\rm red}$. Therefore, the map

$$I(v) \stackrel{\rm def}{=} \langle v\,|z\rangle \tag{2.7}$$

where $v \in \mathcal{H}_{\mathrm{red}}$ and $z \in \mathbb{D}$, is an antilinear and one-to-one map of the Hilbert space $\mathcal{H}_{\mathrm{red}}$ into the vector space $\mathcal{O}(\mathbb{D})$ of holomorphic functions on the disc $\mathbb{D}$. It was shown in [Odz] that the image $I(\mathcal{H}_{\mathrm{red}})$ is isomorphic to the Hilbert space $L^2\mathcal{O}(\mathbb{D}, \mathrm{d}\mu_{\mathcal{R}})$ containing holomorphic functions on $\mathbb{D}$ which are square integrable with respect to the measure

$$\mathrm{d}\mu_{\mathcal{R}}(z,\bar{z}) = \frac{1}{2\pi}\frac{1}{\mathrm{Exp}_{\mathcal{R}}(x)}\frac{1}{(1-q)(q;q)_\infty}\gamma\left(\frac{1}{x}\right)\lim_{a\to\infty}\frac{a^{-\frac{\log x}{\log q}}(a;q)_\infty}{(ax;q)_\infty}\mathrm{d}_q x\,\mathrm{d}\varphi \tag{2.8}$$

where $z = \sqrt{x}\mathrm{e}^{\mathrm{i}\varphi}$, $\mathrm{d}_q x$ is the Jackson measure (see appendix A) and $\mathrm{d}\varphi$ is the Lebesque measure on the circle S^1. The $\mathcal{R}$-exponential function $\mathrm{Exp}_{\mathcal{R}}$ (see [Odz]) and the function γ are defined by

$$\mathrm{Exp}_{\mathcal{R}}(x) := \langle z|z\rangle = \sum_{n=0}^{\infty}\frac{x^n}{\mathcal{R}(q)\ldots\mathcal{R}(q^n)} \tag{2.9}$$

$$\gamma(z) := \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\frac{\mathcal{R}(q;q)_{n-k}}{(q;q)_k}(-1)^k q^{\binom{k}{2}}\right)z^k \tag{2.10}$$

with

$$\mathcal{R}(q;q)_k := \mathcal{R}(q)\mathcal{R}(q^2)\ldots\mathcal{R}(q^k) \tag{2.11}$$

$$(q;q)_k := (1-q)(1-q^2)\ldots(1-q^k) \tag{2.12}$$

$$(x;q)_\infty := \prod_{k=0}^{\infty}\left(1-q^k x\right). \tag{2.13}$$

Let us remark here that the existence of morphism (2.7) is equivalent to the existence of the resolution of unity of the type

$$\int_{\mathbb{D}}|z\rangle\langle z|\,\mathrm{d}\mu_{\mathcal{R}}(z,\bar{z}) = 1. \tag{2.14}$$

By holomorphic representation of $\mathcal{A}_{\mathcal{R}}$ algebra we will understand the representation in the Hilbert space $L^2\mathcal{O}(\mathbb{D}, \mathrm{d}\mu_{\mathcal{R}})$. Straightforward calculation shows that

$$A\varphi(z) = \partial_{\mathcal{R}}\varphi(z) \tag{2.15}$$

$$A^*\varphi(z) = z\varphi(z) \tag{2.16}$$

$$Q\varphi(z) = \varphi(qz) \tag{2.17}$$

$$H_{\mathrm{red}}\varphi(z) = (\mathcal{D}(Q) + z + \partial_{\mathcal{R}})\varphi(z) \tag{2.18}$$

where $\varphi \in L^2\mathcal{O}(\mathbb{D}, \mathrm{d}\mu_{\mathcal{R}})$ and $\partial_{\mathcal{R}}$ is an $\mathcal{R}$-difference operator given by

$$\partial_{\mathcal{R}}\varphi(z) := \mathcal{R}(qQ)\,\partial_0\varphi(z) \tag{2.19}$$

$$\partial_0\varphi(z) := \frac{\varphi(z)-\varphi(0)}{z} \tag{2.20}$$

see [Odz].

In the special case of

$$\mathcal{R}(x) = \frac{1-x}{1-q} \tag{2.21}$$

$\partial_{\mathcal{R}}$ is a q-derivative ∂_q, and the standard derivative $\frac{\mathrm{d}}{\mathrm{d}z}$ is obtained from ∂_q for the limit $q \to 1$. The quantum algebra $\mathcal{A}_{\mathcal{R}}$ of $\mathcal{R}$ given by (2.21) is a q-deformation of the Heisenberg algebra. Hence the analytic realization of $\mathcal{A}_{\mathcal{R}}$ introduced above is a natural generalization of the Bergman–Fock–Segal representation of the Heisenberg algebra.

The second representation of $\mathcal{A}_{\mathcal{R}}$ is related to the spectral measure of a selfadjoint extension of the Hamiltonian (1.43). The action of $H_{\rm red}$ on the elements of the orthonormal basis $\{|n\rangle\}_{n=0}^{\infty}$ is given in terms of a three-diagonal (Jacobi) matrix

$$H_{\rm red}\,|n\rangle = \sqrt{\mathcal{R}\,(q^n)}\,|n-1\rangle + \mathcal{D}\left(q^n\right)|n\rangle + \sqrt{\mathcal{R}(q^{n+1})}\,|n+1\rangle\,. \tag{2.22}$$

We will call this matrix the Jacobi matrix of the operator $H_{\rm red}$.

The operator $H_{\rm red}$ is symmetric and its domain $D_{H_{\rm red}}$ contains all finite linear combinations of the basis elements $\{|n\rangle\}_{n=0}^{\infty}$. The theory of such operators is strictly related to the theory of orthogonal polynomials [A-G, A, Ch, Su].

Let $\mathcal{K}_\omega$ denote the deficiency subspace of $H_{\rm red}$ for $\omega \in \mathbb{C}$ and $\text{Im}\ \omega \neq 0$

$$\mathcal{K}_\omega := \left((H_{\rm red} - \omega 1)\, D_{H_{\rm red}}\right)^{\perp}. \tag{2.23}$$

The deficiency indices (n_+, n_-)

$$n_+ = \dim \mathcal{K}_\omega \qquad \text{for} \quad \text{Im}\ \omega > 0 \tag{2.24}$$

$$n_- = \dim \mathcal{K}_\omega \qquad \text{for} \quad \text{Im}\ \omega < 0 \tag{2.25}$$

of the operator $H_{\rm red}$ are $(0, 0)$ or $(1, 1)$. In order to show this property, one should observe that $|v\rangle \in \mathcal{K}_{\overline{\omega}}$ if and only if

$$H^*_{\rm red}\,|v\rangle = \omega|v\rangle \tag{2.26}$$

where $H^*_{\rm red}$ is the adjoint of $H_{\rm red}$. The vector

$$|v\rangle = \sum_{n=0}^{\infty} P_n\,(\omega)\,|n\rangle \in \mathcal{H}_{\rm red} \tag{2.27}$$

solves (2.26) if and only if the coefficients $P_n\,(\omega)$ satisfy the three-term recurrence equation

$$\omega P_n\,(\omega) = \sqrt{\mathcal{R}(q^n)}\,P_{n-1}\,(\omega) + \mathcal{D}(q^n)\,P_n\,(\omega) + \sqrt{\mathcal{R}(q^{n+1})}\,P_{n+1}\,(\omega) \tag{2.28}$$

$n \in \mathbb{N}$, with the initial conditions

$$P_0\,(\omega) \equiv 1 \qquad P_1\,(\omega) = \frac{\omega - \mathcal{D}\,(1)}{\mathcal{R}\,(q)} \tag{2.29}$$

and

$$\sum_{n=0}^{\infty} |P_n\,(\omega)|^2 < +\infty. \tag{2.30}$$

Hence n_+ and n_- are equal to 0 or 1.

Since every $P_n(\omega)$ is a real polynomial of degree n of the complex variable ω one has $n_+ = n_-$.

Following [A] we will call the Jacobi matrix of $H_{\rm red}$ to be of type D or C if the deficiency indices of $H_{\rm red}$ are $(0, 0)$ or $(1, 1)$, respectively.

Proposition 2.1. *(i) If*

$$\sum_{n=0}^{\infty} \frac{1}{\sqrt{\mathcal{R}\,(q^n)}} = +\infty \tag{2.31}$$

then the operator $H_{\rm red}$ has deficiency indices $(0, 0)$. This is equivalent to its essential selfadjointness.

(ii) If the set of the coherent states $|z\rangle$ of the annihilation operator A is parametrized by the disc $\mathbb{D}$ of finite radius $\mathcal{R}\,(0) < +\infty$ then $H_{\rm red}$ is essentially selfadjoint.

(iii) If the deficiency indices of $H_{\rm red}$ are $(1, 1)$ then the coherent states $|z\rangle$ of A exist for any $z \in \mathbb{C}$.

Proof.

(i) Let $\{Q_n(\omega)\}_{n=0}^{\infty}$ be another solution of the recurrence relation (2.28) with the initial conditions

$$Q_0(\omega) \equiv 0 \qquad Q_1(\omega) \equiv \frac{1}{\sqrt{\mathcal{R}(q)}}. \tag{2.32}$$

Then one has

$$P_{k-1}(\omega) Q_k(\omega) - P_k(\omega) Q_{k-1}(\omega) = \frac{1}{\sqrt{\mathcal{R}\left(q^k\right)}} \tag{2.33}$$

where $k \in \mathbb{N}$. Applying the Schwartz inequality to (2.33), one finds

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{\mathcal{R}(q^n)}} \leqslant 2\left(\sum_{n=0}^{\infty} |P_n(\omega)|^2\right)\left(\sum_{n=0}^{\infty} |Q_n(\omega)|^2\right). \tag{2.34}$$

Moreover one can prove (see [A]) that if $\{P_n(\omega)\}_{n=0}^{\infty}$ satisfies (2.30), then $\{Q_n(\omega)\}_{n=0}^{\infty}$ also satisfies it. Therefore $\sum_{n=0}^{\infty} \frac{1}{\sqrt{\mathcal{R}(q^n)}} < +\infty$, which contradicts (2.31). Thus, one has

$$\sum_{n=0}^{\infty} |P_n(\omega)|^2 = \infty \tag{2.35}$$

meaning that $n_+ = n_- = 0$. We apply Th.VIII.3 from [R-S] vol 1.

(ii) If $\mathcal{R}(0) < +\infty$ then condition (2.35) follows immediately from (2.31) and statement (ii) follows from statement (i).

(iii) One proves it *ad absurdum* using the previous statement.

□

Let $\hat{H}_{\text{red}}$ be some selfadjoint extension of H_{red}. In the case of type C such extensions are parametrized by points of S^1, see [R-S], whereas in the case of type D the extension is unique. Let $\mathrm{d}E_{\mathcal{R},D}(\omega)$ be the spectral measure of $\hat{H}_{\text{red}}$.

By the spectral representation of $\mathcal{A}_R$ we will call the representation in the space $L^2\left(\mathbb{R}, \mathrm{d}\sigma_{\mathcal{R},D}\right)$ where the measure $\mathrm{d}\sigma_{\mathcal{R},D}$ is given by

$$\mathrm{d}\sigma_{\mathcal{R},D}(\omega) := \left\langle 0|\mathrm{d}E_{\mathcal{R},D}(\omega)0\right\rangle \tag{2.36}$$

and the operator $\hat{H}_{\text{red}}$ acts by multiplication with the identity function on $\mathbb{R}$. Let $U_{PN} : \mathcal{H}_{\text{red}} \longrightarrow L^2\left(\mathbb{R}, \mathrm{d}\sigma_{\mathcal{R},D}\right)$ denote the intertwining operator for these two representations. Since the polynomials $\{P_n(\omega)\}_{n=0}^{\infty}$ form an orthonormal basis in $L^2\left(\mathbb{R}, \mathrm{d}\sigma_{\mathcal{R},D}\right)$, it is convenient to write the intertwining operator using the physical notation of Dirac

$$U_{PN} = \sum_{n=0}^{\infty} P_n(\omega) \otimes \langle n| . \tag{2.37}$$

The convergence here is understood in the sense of weak topology. The relation of the holomorphic representation and the spectral representation is given by the isomorphism of Hilbert spaces

$$U_{PZ} : L^2\mathcal{O}(\mathbb{D}, \mathrm{d}\mu_{\mathcal{R}}) \to L^2\left(\mathbb{R}, \mathrm{d}\sigma_{\mathcal{R},D}\right) \tag{2.38}$$

and U_{PZ} which can be written as

$$U_{PZ} = \sum_{n=0}^{\infty} P_n(\omega) \otimes \frac{z^n}{\sqrt{\mathcal{R}(q) \ldots \mathcal{R}(q^n)}} \tag{2.39}$$

where $\left\{ \frac{z^n}{\sqrt{\mathcal{R}(q)\ldots\mathcal{R}(q^n)}} \right\}_{n=0}^{\infty}$ is an orthonormal basis in $L^2\mathcal{O}(\mathbb{D}, \mathrm{d}\mu_{\mathcal{R}})$. Similarly let

$$U_{ZN} : \mathcal{H}_{\mathrm{red}} \longrightarrow L^2(\mathbb{D}, \mathrm{d}\mu_{\mathcal{R}}) \tag{2.40}$$

be given by

$$U_{ZN} := \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{\mathcal{R}(q)\ldots\mathcal{R}(q^n)}} \otimes \langle n| . \tag{2.41}$$

We thus get the following commutative diagram of Hilbert space isomorphisms:

$$\begin{array}{ccc} & \mathcal{H}_{\mathrm{red}} & \\ \overset{U_{PZ}}{\swarrow} & & \overset{U_{ZN}}{\searrow} \\ L^2\left(\mathbb{R}, \mathrm{d}\sigma_{\mathcal{R},D}\right) & \overset{U_{PN}}{\longleftarrow} & L^2\mathcal{O}(\mathbb{D}, \mathrm{d}\mu_{\mathcal{R}}) . \end{array} \tag{2.42}$$

If the series

$$V(\omega, z) := \sum_{n=0}^{\infty} P_n(\omega) \frac{z^n}{\sqrt{\mathcal{R}(q)\ldots\mathcal{R}(q^n)}} \tag{2.43}$$

is pointwise convergent for all $\omega \in [a, b]$ and $z \in \mathbb{D}$, the isomorphism U_{PZ} can be represented as the integral transform

$$(U_{PZ}\varphi)(\omega) = \int_{\mathbb{D}} V(\omega, z)\, \varphi(z)\; \mathrm{d}\mu_{\mathcal{R}}\;(z, \bar{z}) . \tag{2.44}$$

The kernel $V(\omega, z)$ of this transform satisfies the $\mathcal{R}$-difference equation

$$(\omega - z)\, V(\omega, z) = \mathcal{D}(Q)\, V(\omega, z) + \partial_{\mathcal{R}} V(\omega, z) . \tag{2.45}$$

Of course, from a orthogonal polynomial theory point of view $V(\omega, z)$ is nothing more than the generating function for the family of orthogonal polynomials under consideration.

The function $V(\omega, z)$ is useful for calculating many important physical quantities of the system described by the Hamiltonian H_{red}.

First of all, note that using (2.44) and (2.45) one obtains

$$\langle v \left| H_{\mathrm{red}}^n z \right\rangle = \int_{\mathbb{R}} \overline{V(\omega, v)} \omega^n V(\omega, z)\; \mathrm{d}\sigma_{\mathcal{R},D}(\omega) = (z + \mathcal{D}(Q) + \partial_{\mathcal{R}})^n\, \mathrm{Exp}_{\mathcal{R}}(\bar{v}z) . \tag{2.46}$$

In particular putting $n = 0$ gives

$$\mathrm{Exp}_{\mathcal{R}}(\bar{v}z) = \int_{\mathbb{R}} \overline{V(\omega, v)} V(\omega, z)\; \mathrm{d}\sigma_{\mathcal{R},D}(\omega) \tag{2.47}$$

which is the integral representation of the $\mathcal{R}$-exponential function $\mathrm{Exp}_{\mathcal{R}}(\bar{v}z)$. This function satisfies the equation

$$\partial_{\mathcal{R}} \mathrm{Exp}_{\mathcal{R}}(\bar{v}z) = \bar{v}\, \mathrm{Exp}_{\mathcal{R}}(\bar{v}z) . \tag{2.48}$$

Let us recall that $\mathrm{Exp}_{\mathcal{R}}(\bar{v}\cdot) \in L^2\mathcal{O}(\mathbb{D}, \mathrm{d}\mu_{\mathcal{R}})$ is the expression of the coherent state holomorphically represented.

The evolution operator $U(t) := \mathrm{e}^{\mathrm{i}\hat{H}_{\mathrm{red}} t}$ acts on the function from $L^2\left(\mathbb{R}, \mathrm{d}\sigma_{\mathcal{R},D}\right)$ as multiplication by the phase factors

$$U(t)\, \psi(\omega) = \mathrm{e}^{\mathrm{i}\omega t} \psi(\omega) \tag{2.49}$$

and enables us to calculate the transition amplitudes between coherent states

$$\langle v | U(t)\, z \rangle = \int_{\mathbb{R}} \overline{V(\omega, v)} \mathrm{e}^{\mathrm{i}\omega t} V(\omega, v)\; \mathrm{d}\sigma_{\mathcal{R},D}(\omega). \tag{2.50}$$

The vacuum–vacuum transition amplitude is also important for physicists and is given by

$$\langle 0|U\ (t)\, 0\rangle = \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\omega t}\, \mathrm{d}\sigma_{\mathcal{R},D}(\omega) = \sum_{n=0}^{\infty} \frac{(\mathrm{i}t)^n}{n!} \mu_n \tag{2.51}$$

where μ_n is the nth moment of the measure $\mathrm{d}\sigma_{\mathcal{R},D}$.

All the above shows that the measure $\mathrm{d}\sigma_{\mathcal{R},D}$ plays a significant role in the description of our physical system. The construction of $\mathrm{d}\sigma_{\mathcal{R},D}$ is one of the most important problems which has to be solved in order to recover the dynamics. In order to give an example (see [A]) of the solution of this problem, let us recall the notion of a simple symmetric operator.

The Hilbert subspace $\mathcal{H}_1 \subset \mathcal{H}$ is a reducible subspace of the linear operator $T : \mathcal{H} \longrightarrow \mathcal{H}$ if $\mathcal{H}_1$ and $\mathcal{H}_2 := \mathcal{H}_1^{\perp}$ are invariant subspaces for T and for the orthogonal projection $\Pi_1 : \mathcal{H} \longrightarrow \mathcal{H}_1$ one has $\Pi_1\,(D_T) \subset D_T$. The symmetric operator T is simple if there does not exist an irreducible subspace of T such that $T_{|\mathcal{H}_1}$ has a selfadjoint extension in $\mathcal{H}_1$.

If the Jacobi matrix of the reduced Hamiltonian H_{red} is of type C then the series

$$\sum_{n=0}^{\infty} |P_n(\omega)|^2 \qquad \text{and} \qquad \sum_{n=0}^{\infty} |Q_n\ (\omega)|^2 \tag{2.52}$$

are almost uniformly convergent on $\mathbb{C}$. Thus the functions $A(\omega)$, $B(\omega)$, $C(\omega)$ and $D(\omega)$ defined by

$$\begin{aligned} A(\omega) &= \omega \sum_{k=0}^{\infty} Q_k\ (0)\ Q_k(\omega) \\ B(\omega) &= -1 + \omega \sum_{k=0}^{\infty} Q_k\ (0)\ P_k(\omega) \\ C(\omega) &= 1 + \omega \sum_{k=0}^{\infty} P_k\ (0)\ Q_k(\omega) \\ D(\omega) &= \omega \sum_{k=0}^{\infty} P_k\ (0)\ P_k(\omega) \end{aligned} \tag{2.53}$$

are entire functions. If in addition H_{red} is a simple and closed operator then the spectral measure $\mathrm{d}E_{\mathcal{R},D}(\omega)$ of its arbitrary selfadjoint extension $\hat{H}_{\mathrm{red}}$ is localized at the zeros, ω_i, $i = 1, 2, \ldots,$ of the function $q(\omega) = B(\omega)t - D(\omega)$ for some $t \in \mathbb{R}$. The steps μ_i, $i = 1, 2, \ldots$ of the measure $\mathrm{d}\sigma_{\mathcal{R},D}(\omega) := \langle 0|\mathrm{d}E_{\mathcal{R},D}(\omega)0\rangle$ satisfy the following conditions:

$$\sum_{i=1}^{\infty} \frac{1}{\mu_i \left(1+\omega_i^2\right) |q'\,(\omega_i)|^2} < \infty \tag{2.54}$$

$$\sum_{i=1}^{\infty} \frac{1}{\mu_i\, |q'\,(\omega_i)|^2} = \infty. \tag{2.55}$$

From the identity

$$|n\rangle = P_n\,(H_{\mathrm{red}})\,|0\rangle \tag{2.56}$$

it follows that if the Jacobi matrix of H_{red} is of type D then H_{red} has a simple spectrum. Conversely, one can prove (see [St]) that every selfadjoint operator $H = H^*$ with a simple spectrum may be represented in some orthonormal basis by the formula (2.22) where the Jacobi matrix is of the type D.

Using (2.3)–(2.5) we can associate with it $\mathcal{A}_R$ algebra with a suitable structural function $\mathcal{R}$. This fact indicates that algebras of this kind are important tools for investigating symmetry structures of physical systems with dynamics generated by Hamiltonians with a simple spectrum.

In the next section we will describe the situation when neither coherent states nor the kernel $V(\omega, z)$ do exist.

3. Integrable systems related to q-Hahn class polynomials

We will integrate the quantum systems related to $\mathcal{A}_{\mathcal{R}}$ and Hamiltonian $H_{\rm red} =: H_{AB}$ algebras with structural functions of the form

$$\mathcal{R}_{AB}(x) = (1-q)\,x\left[\partial_q \eta(x) - \beta(x)\,\partial_q \beta(x)\right] \tag{3.1}$$

$$\mathcal{D}_{AB}(x) = (1-q)\,x\,\partial_q \beta(x) \tag{3.2}$$

where

$$\beta(x) = \frac{q(1-x)\left[(a_0(1-q)-b_1)x + b_1 q\right]}{(1-q)\left[(a_1(1-q)-b_2)x^2 + b_2 q^2\right]} \tag{3.3}$$

$$\eta(x) = \left\{\frac{\left[(a_0(1-q)-b_1)x + b_1 q^2\right]\left[(a_0(1-q)-b_1)x + b_1 q\right]}{\left[(a_1(1-q)-b_2)x^2 + b_2 q^2\right]\left[(a_1(1-q)-b_2)x^2 + b_2 q^3\right]} + \frac{b_0(1-q)}{(a_1(1-q)-b_2)x^2 + b_2 q^3}\right\}\frac{q^3(1-x)\left(1-q^{-1}x\right)}{(1-q)^2(1+q)}. \tag{3.4}$$

These functions depend on five real parameters a_0, a_1, b_0, b_1 and b_2. It will be shown later that it is natural to introduce the following two polynomials:

$$A(\omega) = a_1\omega + a_0 \tag{3.5}$$

$$B(\omega) = b_2\omega^2 + b_1\omega + b_0. \tag{3.6}$$

It is clear from (3.3), (3.4) that the pairs of polynomials $(A(\omega), B(\omega))$ of degree one and two, respectively, taken up to the common overall real factor $c \neq 0$, parametrize the models under consideration. The only condition we will impose on $(A(\omega), B(\omega))$ is the one given by $\mathcal{R}_{AB}(q^n) > 0$ for any $n \in \mathbb{N}$.

Analogous to the theory of classical orthogonal polynomials is a q-difference equation (an analogue of Pearson's equation [Su])

$$\partial_q(\varrho B)(\omega) = (\varrho A)(\omega) \tag{3.7}$$

associated with $(A(\omega), B(\omega))$.

We will look for the solutions $\varrho(\omega)$ of (3.7) which satisfy the boundary conditions

$$\varrho(a)\,B(a) = \varrho(b)\,B(b) = 0 \tag{3.8}$$

for some fixed a, b such that $-\infty \leqslant a < b \leqslant \infty$. We thus have the so-called Pearson data $(A(\omega), B(\omega))$ on the interval $(a, b) \subset \mathbb{R}$.

Proposition 3.1. *Let $\varrho(\omega)$ be the solution of the q-Pearson equation (3.7) defined by $(A(\omega), B(\omega))$ and satisfying (3.8). Then $\varrho^{(k)}(\omega) := \varrho\left(q^k\omega\right)B(q\omega)\ldots B\left(q^k\omega\right)$ is the solution of Pearson q-equation (3.7) associated with the pair*

$$A^{(k)}(\omega) := q^k A(q^k\omega) + \frac{1-q^kQ^k}{1-q}\partial_q B(\omega) \tag{3.9}$$

$$B^{(k)}(\omega) := B(\omega) \tag{3.10}$$

where $k \in \mathbb{N}$. If $\varrho(\omega)$ satisfies the boundary conditions (3.8) then $\varrho^{(k)}(\omega)$ also satisfies (3.8).

Proof. By straightforward calculation. □

Let $L^2\left([a,b]\,,\,\mathrm{d}\sigma_{AB}\right)$ be the Hilbert space of square integrable functions with respect to the measure

$$\mathrm{d}\sigma_{AB}(\omega) = \varrho\,(\omega)\;\mathrm{d}_q\omega \tag{3.11}$$

where

$$\mathrm{d}_q\omega = \sum_{k=0}^{\infty}(1-q)\,q^k\big[b\delta\big(\omega - q^k b\big) - a\delta\big(\omega - q^k a\big)\big]\,\mathrm{d}\omega \tag{3.12}$$

is the Jackson measure on the interval $[a,b]$. It will be assumed that the weight function $\varrho\,(\omega)$ satisfies the Pearson q-equation (3.7) supplemented with the boundary condition (3.8).

On applying the orthonormalization procedure to the monomials $\{\omega^n\}_{n=0}^{\infty} \subset L^2\left([a,b]\,,\,\mathrm{d}\sigma_{AB}\right)$ we obtain the system of orthonormal polynomials $\{P_n\,(\omega)\}_{n=0}^{\infty}$

$$\int_a^b P_n(\omega)P_m(\omega)\,\mathrm{d}\sigma_{AB}(\omega) = \delta_{nm} \tag{3.13}$$

which is uniquely determined by the pair $(A\,(\omega)\,,\,B(\omega))$ on the interval $[a,b]$. The polynomials $\{P_n(\omega)\}_{n=0}^{\infty}$ are called q-Hahn class polynomials, see [G-R,H]. Let us denote by $\left\{\tilde{P}_n(\omega)\right\}_{n=0}^{\infty}$ the monic orthogonal polynomial system (OPS) associated with $\{P_n(\omega)\}_{n=0}^{\infty}$ (i.e. $\tilde{P}_n(\omega) = \frac{1}{\alpha_n}P_n(\omega)$ where α_n is the coefficient of the highest power in $P_n(\omega)$).

Theorem 3.1. *If* $\left\{\tilde{P}_n(\omega)\right\}_{n=0}^{\infty}$ *is the monic OPS corresponding to the Pearson data* $(A(\omega),\,B(\omega))$ *then the family of polynomials*

$$\left\{\frac{1}{[n]\,[n-1]\ldots[n-k]}\partial_q^k\tilde{P}_n(\omega)\right\}_{n=0}^{\infty} \tag{3.14}$$

where

$$[k] := \frac{1-q^k}{1-q}$$

forms the monic OPS corresponding to $\left(A^{(k)}(\omega),\,B^{(k)}\,(\omega)\right)$ *with the same boundary conditions (3.8).*

Proof. For $k \leqslant n-2$ we have

$$\int_a^b \tilde{P}_n(\omega)\omega^k A(\omega)\varrho(\omega)\,\mathrm{d}_q\omega = 0. \tag{3.15}$$

Using Leibnitz's rule, (3.7) and (3.8) we obtain

$$\begin{aligned}
0 = \int_a^b & \tilde{P}_n(\omega)\omega^k\left(\partial_q B\varrho\right)(\omega)\,\mathrm{d}_q\omega \\
&= \tilde{P}_n(\omega)\omega^k B\,(\omega)\,\varrho(\omega)|_a^b - \int_a^b \partial_q\big(\tilde{P}_n(\omega)\omega^k\big)B\,(q\omega)\,\varrho\,(q\omega)\;\mathrm{d}_q\omega \\
&= -\int_a^b \partial_q\big(\tilde{P}_n(\omega)\omega^k\big)\,(B\,(\omega) - (1-q)\,\omega A(\omega))\,\varrho(\omega)\,\mathrm{d}_q\omega \\
&= -\int_a^b \partial_q\tilde{P}_n(\omega)\,(q\omega)^k\,(B(\omega) - (1-q)\,\omega A\,(\omega))\,\varrho(\omega)\,\mathrm{d}_q\omega \\
&\quad -\int_a^b \tilde{P}_n(\omega)\,[k]\,\omega^{k-1}\,(B(\omega) - (1-q)\,\omega A(\omega))\,\varrho\,(\omega)\;\mathrm{d}_q\omega.
\end{aligned}$$

The degree of the polynomial

$$[k]\,\omega^{k-1}\,(B(\omega) - (1-q)\,\omega A(\omega)) \tag{3.16}$$

is $k+1 < n$, and from (3.9), (3.10) one finds that

$$\int_a^b \partial_q \tilde{P}_n(\omega)\omega^k \varrho^{(1)}(\omega)\,\mathrm{d}_q\omega = 0 \tag{3.17}$$

for $k \leqslant n-2$. This shows that the polynomials $\left\{\frac{1}{[n]}\partial_q \tilde{P}_n(\omega)\right\}_{n=1}^{\infty}$ form a monic OPS for the Pearson data $\left(A^{(1)}(\omega), B^{(1)}(\omega)\right)$ given by (3.9) and (3.10). □

Since, for the rational function $\mathcal{R}_{AB}(x)$ of (3.1) one has $\mathcal{R}_{AB}(0) = 0$ proposition 2.1 and it's consequences as described in the previous section imply that for the case under consideration the following statements are true:

(i) the operator $\overline{H}_{AB}$ is selfadjoint and has a simple spectrum;
(ii) the coherent states do not exist for $\mathcal{A}_{\mathcal{R}_{AB}}$;
(iii) the Hilbert space $\mathcal{H}_{\mathrm{red}}$ is unitarily isomorphic to $L^2([a,b], \mathrm{d}\sigma_{AB})$, with the isomorphism given by (2.37).

Hence, the Hahn class polynomials $\{P_n(\omega)\}_{n=0}^{\infty}$ form an orthonormal basis in $L^2([a,b], \mathrm{d}\sigma_{AB})$. The measure $\mathrm{d}\sigma_{AB}$ is the expected value of the spectral measure $\mathrm{d}E_{AB}$ in the vacuum state $|0\rangle$.

It is thus clear that the properties of Hahn class polynomials are crucial in obtaining a better understanding of the physical systems corresponding to $\mathcal{A}_{\mathcal{R}_{AB}}$ algebra. The following theorem describes some of the important properties of these polynomials.

Theorem 3.2 (Hahn). *Fix some Pearson data $(A(\omega), B(\omega))$ on the interval $[a,b] \subset \mathbb{R}$. Then the following statements are equivalent.*

(A) The family of polynomials $\left\{\tilde{P}_n(\omega)\right\}_{n=0}^{\infty}$ forms the monic OPS with respect to $(A(\omega), B(\omega))$.

(B) The polynomials are given by the Rodriques' formula

$$\tilde{P}_n(\omega) = c_n \frac{1}{\varrho(\omega)} \partial_q^n\left[\varrho(\omega)B(\omega)B\left(q^{-1}\omega\right)\ldots B\left(q^{-(n-1)}\omega\right)\right] \tag{3.18}$$

$n \in \mathbb{N}$, where c_n is a normalization constant.

(C) The polynomials $\left\{\tilde{P}_n(\omega)\right\}_{n=0}^{\infty}$ satisfy the following q-difference equation (Hahn equation):

$$\left(A(\omega)\partial_q + B(\omega)\,\partial_q Q^{-1}\partial_q\right)\tilde{P}_n(\omega) = \lambda_n \tilde{P}_n(\omega) \tag{3.19}$$

where

$$\begin{aligned} \lambda_n &= a_1[n] + b_2[n][n-1]\,q^{-(n-1)} \qquad n = 2, 3, \ldots \\ \lambda_1 &= a_1. \end{aligned} \tag{3.20}$$

(D) Every polynomial of the system $\left\{\tilde{P}_n(\omega)\right\}_{n=0}^{\infty}$ is given by

$$\begin{aligned} \tilde{P}_n(\omega) = \prod_{k=0}^{n-1} &\frac{1}{a_1^{(k)} - b_2[-n+1+k]} \\ &\times\left(A^{(0)}(\omega) + B(\omega)\partial_q Q^{-1}\right)\ldots\left(A^{(k-1)}(\omega) + B(\omega)\,\partial_q Q^{-1}\right)\cdot 1 \end{aligned} \tag{3.21}$$

where the linear functions

$$A^{(k)}(\omega) = a_1^{(k)}\omega + a_0^{(k)} \tag{3.22}$$

are defined in (3.9).

(E) *The polynomials of the system* $\left\{\tilde{P}_n(\omega)\right\}_{n=0}^{\infty}$ *are related by the three-term recurrence formula*

$$\tilde{P}_{n+1}(\omega) + \mathcal{R}_{AB}\left(q^n\right)\tilde{P}_{n-1}(\omega) = \left(\omega - \mathcal{D}_{AB}\left(q^n\right)\right)\tilde{P}_n(\omega) \tag{3.23}$$

with the initial condition $P_0(\omega) \equiv 1$.

The proofs of the equivalence of A, B, C may be found in the original paper of Hahn [H]. The recurrence formula (3.23) is considered there without specification of the form of the structural functions $\mathcal{R}_{AB}, \mathcal{D}_{AB}$. The only assumption made is that they are rational functions of the parameter q. A complete proof of this theorem is given in appendix B.

From section 2 it follows that the problem of integration of the multi-boson system described in $L^2([a, b], \mathrm{d}\sigma_{AB})$ is reduced to the construction of the measure $\mathrm{d}\sigma_{AB}$. According to (3.11) the measure $\mathrm{d}\sigma_{AB}$ is given by the density function $\varrho(\omega)$ which is a solution of the q-difference Pearson equation (3.7). Let us therefore present all possible solutions of (3.7) from the class of meromorphic functions. Using (3.5) and (3.6) we can rewrite (3.7) in the form

$$\begin{aligned}\varrho(\omega) &= \frac{B(q\omega)}{B(\omega) - (1-q)\omega A(\omega)}\varrho(q\omega) \\ &= \frac{b_2q^2\omega^2 + b_1q\omega + b_0}{(b_2 - (1-q)a_1)\omega^2 + (b_1 - (1-q)a_0)\omega + b_0}\varrho(q\omega)\end{aligned} \tag{3.24}$$

and after standard calculations we obtain the classes of solutions detailed in Proposition 3.2 depending on the values of the parameters b_2, b_1, b_0, a_1, a_0.

Proposition 3.2. *One has the following subcases of the solutions of the q-difference Pearson equation (3.7):*

(i) *If* $b_0 \neq 0$ *and* $b_2 - (1-q)a_1 \neq 0$, *then*

$$\varrho(\omega) = \frac{\left(\frac{q\omega}{a}; q\right)_\infty \left(\frac{q\omega}{b}; q\right)_\infty}{\left(\frac{\omega}{c}; q\right)_\infty \left(\frac{\omega}{\mathrm{d}}; q\right)_\infty} \tag{3.25}$$

where $a \neq 0$, $b \neq 0$ *are the roots of the polynomial* $B(\omega)$ *and* $c \neq 0$, $d \neq 0$ *are the roots of the polynomial* $B(\omega) - (1-q)\omega A(\omega)$.

(ii) *If* $b_0 \neq 0$ *and* $b_1 - (1-q)a_0 \neq 0$ *and* $b_2 - (1-q)a_1 = 0$, *then*

$$\varrho(\omega) = \frac{\left(\frac{q\omega}{a}; q\right)_\infty \left(\frac{q\omega}{b}; q\right)_\infty}{\left(\frac{\omega}{c}; q\right)_\infty} \tag{3.26}$$

where $a \neq 0$, $b \neq 0$ *are roots of the polynomial* $B(\omega)$ *and* $c \neq 0$ *is the root of the polynomial* $B(\omega) - (1-q)\omega A(\omega)$.

(iii) *If* $b_0 \neq 0$ *and* $b_1 - (1-q)a_0 = 0$ *and* $b_2 - (1-q)a_1 = 0$, *then*

$$\varrho(\omega) = \left(\frac{q\omega}{a}; q\right)_\infty \left(\frac{q\omega}{b}; q\right)_\infty \tag{3.27}$$

where $a \neq 0$, $b \neq 0$ *are roots of the polynomial* $B(\omega)$.

(iv) *If* $b_0 = 0$, $b_1 \neq 0$ *and* $b_1 - (1-q)a_0 \neq 0$ *and* $b_2 - (1-q)a_1 \neq 0$ *and* $b_2 \neq 0$, *then*

$$\varrho(\omega) = \omega^r \frac{(\frac{q\omega}{a}; q)_\infty}{(\frac{\omega}{c}; q)_\infty} \tag{3.28}$$

where $a \neq 0$ *is the root of the polynomial* $B(\omega)$, $c \neq 0$ *is the root of the polynomial* $B(\omega) - (1-q)\omega A(\omega)$ *and* $q^{-r} = \left|\frac{qb_1}{b_1 - (1-q)a_0}\right|$.

(v) If $b_0 = 0$, $b_1 \neq 0$ *and* $b_1 - (1-q)\,a_0 \neq 0$ *and* $b_2 - (1-q)\,a_1 = 0$ *and* $b_2 \neq 0$, *then*

$$\varrho(\omega) = \omega^r \left(\frac{q\omega}{a}; q\right)_\infty \tag{3.29}$$

where $a \neq 0$ *is the root of the polynomial* $B(\omega)$ *and* $q^{-r} = \left|\frac{qb_1}{b_1-(1-q)a_0}\right|$.

(vi) If $b_0 = b_1 - (1-q)\,a_0 = 0$, $b_1 \neq 0$, $b_2 \neq 0$ *and* $b_2 - (1-q)a_1 \neq 0$, *then*

(*a*)
$$\varrho(\omega) = \omega^r \frac{\left(\frac{q\omega}{a}; q\right)_\infty}{(-\omega; q)_\infty(-q\omega^{-1}; q)_\infty} \tag{3.30}$$

for $q^{-r} = \frac{qb_1}{b_2-(1-q)a_1} > 0$;

(*b*)
$$\varrho(\omega) = \omega^r \frac{\left(\frac{q\omega}{a}; q\right)_\infty}{(\omega; q)_\infty(q\omega^{-1}; q)_\infty} \tag{3.31}$$

for $-q^{-r} = \frac{qb_1}{b_2-(1-q)a_1} < 0$, *where* $a \neq 0$ *is the root of the polynomial* $B(\omega)$.

(vii) If $b_0 = b_1 = 0$, $b_1 - (1-q)a_0 \neq 0$ *and* $b_2 \neq 0$, *then*

(*a*)
$$\varrho(\omega) = \omega^r \frac{(-\omega; q)_\infty(-q\omega^{-1}; q)_\infty}{\left(\frac{\omega}{c}; q\right)_\infty} \tag{3.32}$$

for $q^{-r} = \frac{q^2b_2}{b_1-(1-q)a_0} > 0$;

(*b*)
$$\varrho(\omega) = \omega^r \frac{(\omega; q)_\infty(q\omega^{-1}; q)_\infty}{\left(\frac{\omega}{c}; q\right)_\infty} \tag{3.33}$$

for $-q^{-r} = \frac{q^2b_2}{b_1-(1-q)a_0} < 0$,
where $c \neq 0$ *is the root of the polynomial* $B(\omega) - (1-q)\omega A(\omega)$.

(viii) If $b_0 = b_1 = b_1 - (1-q)\,a_0 = 0$, *and* $b_2 - (1-q)a_1 \neq 0$ *and* $b_2 \neq 0$, *then*

$$\varrho(\omega) = \omega^r \tag{3.34}$$

for $q^{-r} = \left|\frac{q^2b_2}{b_2-(1-q)a_1}\right|$.

Proof. The subcases (i)–(iii) are easily obtained by iteration. The points (iv)–(viii) are proved by calculation of the Laurent expansion coefficient and application of Ramanujan's identities (see [G-R]). □

We can now determine the interval of integration in (3.13) and determine the conditions on polynomials $A(\omega)$ and $B(\omega)$ such that the measure $\mathrm{d}\sigma_{AB}$ is positive (i.e. $R(q^n) > 0$ for $n \in \mathbb{N}$). It will be convenient to express the conditions on $A(\omega)$ and $B(\omega)$ in terms of the roots of the polynomials $B(\omega)$ and $B(\omega) - (1-q)\omega A(\omega)$.

Proposition 3.3. *The measure* $d\sigma_{AB}$ *is positive and the condition (3.8) is fulfilled if and only if (in the notation and classification of proposition 3.2):*

(i) The integration interval is $[a, b]$ *with* $a < 0 < b$ *and* c, d *satisfies one of the following conditions:*
- (α) $c = \overline{d}$,
- (β) $c < a$ *and* $d > b$,
- (γ) $c, d < a$,
- (δ) *there exists* $K \in \mathbb{N}$ *such that* $q^{K-1}a < c, d < q^K a$,
- (ϵ) $c, d > b$,
- (ζ) *there exists* $K \in \mathbb{N}$ *such that* $q^K b < c, d < q^{K-1} b$.

(ii) The integration interval is $[a, b]$ *with* $a < 0 < b$ *and* $c < a$ *or* $c > b$.

(iii) The integration interval is $[a, b]$ *with* $a < 0 < b$.

(iv) This case splits into two subcases:

(1) For $a > 0$ *the integration interval is* $[0, a]$ *and* $c < 0$ *or* $c > a$.

(2) For $a < 0$ *the integration interval is* $[a, 0]$ *and* $c < a$ *or* $c > 0$ *and* r *has to be such that* $a^r > 0$.

(v) This case splits into two subcases:

(1) For $a > 0$ *the integration interval is* $[0, a]$.

(2) For $a < 0$ *the integration interval is* $[a, 0]$ *and* r *has to be such that* $a^r > 0$.

(vi) This case splits into two subcases:

(1) For $a > 0$ *the integration interval is* $[0, a]$.

(2) For $a < 0$ *the integration interval is* $[a, 0]$ *and* r *have to be such that* $a^r > 0$.

(vii) In this case $\mathcal{R}_{AB}(q^n)$ *are not positive for large enough* n.

(viii) In this case $\mathcal{R}_{AB}(q^n) = 0$ *and* $\mathcal{D}_{AB}(q^n) = 0$ *for* $n \in \mathbb{N}$.

Proof. (i) The equation $B(\omega)\varrho(\omega) = 0$ is solved by aq^{-k+1} and by bq^{-k+1} for $k \in \mathbb{N}$. For any function $f(\omega)$ and any $k, l \in \mathbb{N}$, using (3.11) and (3.12) one can obtain

$$\int_{aq^{-l+1}}^{aq^{-k+1}} f(\omega)\, \mathrm{d}\sigma_{AB}(\omega) = 0 = \int_{bq^{-l+1}}^{bq^{-k+1}} f(\omega)\, \mathrm{d}\sigma_{AB}(\omega) \tag{3.35}$$

and

$$\int_a^b f(\omega)\, \mathrm{d}\sigma_{AB}(\omega) = \int_{aq^{-k+1}}^{bq^{-l+1}} f(\omega)\, \mathrm{d}\sigma_{AB}(\omega). \tag{3.36}$$

Hence the integration interval is $[a, b]$. The condition of positivity of $\mathrm{d}\sigma_{AB}(\omega)$

$$\int_a^b f(\omega)\, \mathrm{d}\sigma_{AB}(\omega) > 0 \qquad \text{for} \quad f > 0 \tag{3.37}$$

is equivalent to

$$a\varrho(q^i a) < 0 \qquad \text{and} \qquad b\varrho(q^i b) > 0 \qquad \text{for} \quad i = 0, 1, \ldots. \tag{3.38}$$

The continuity of ϱ at $\omega = 0$ gives $a < 0 < b$ and the inequalities

$$\varrho(q^i a) > 0 \qquad \varrho(q^i b) > 0 \qquad \text{for} \quad i = 0, 1, \ldots \tag{3.39}$$

which are solved by (α)–(ζ). The proofs of (ii)–(viii) are similar to the one above. □

The above class of orthogonal polynomials, which we call the q-Hahn class polynomials, contains, as a special case the families of orthogonal polynomials which are well known in the literature. Using a very good paper [K-S] we obtain the following identification:

(1) Putting in (i) (β) $d = 1$ we have the big q-Jacobi polynomials. If in addition we put $b = q$ and $c = \frac{a}{q}$ we obtain the big q-Legendre polynomials.
(2) Putting in (ii) $b < 1$ and $c = 1$ we obtain the big q-Laguerre polynomials.
(3) Putting in (iii) $b = 1$ we obtain the Al-Salam–Carlitz I polynomials. If in addition we assume $a = -1$ we obtain the Discrete q-Hermite I polynomials.
(4) Putting in (iv) (1) $a = 1$ we obtain the little q-Jacobi polynomials. If in addition we put $c = \frac{1}{q}$ and $r = 0$ we obtain the little q-Legendre polynomials.
(5) Putting in (v) (1) $a = 1$ we obtain the little q-Laguerre/Wall polynomials.
(6) Putting in (vi) (1) $r = 0$ we obtain the Alternative q-Charlier polynomials.

We will now find the equations for the moments

$$\mu_n = \int_a^b \omega^n \, \mathrm{d}\sigma_{AB}(\omega) \tag{3.40}$$

of the measure $\mathrm{d}\sigma_{AB}$.

From section 2 it is clear that once the moments are known one may determine many important physical characteristic of the system under consideration.

Multiplying the q-difference Pearson equation (3.7) by $\omega^n q^n$ and using the Leibnitz rule for the q-derivative we obtain the following three-term recurrence equation:

$$-[n]\,(b_2\mu_{n+1} + b_1\mu_n + b_0\mu_{n-1}) = q^n\,(a_1\mu_{n+1} + a_0\mu_n) \tag{3.41}$$

for $n \geqslant 1$, and

$$a_1\mu_1 + a_0\mu_0 = 0. \tag{3.42}$$

The initial rule $\mu_0 = \int_a^b \mathrm{d}\sigma_{AB}(\omega)$ for this recurrence can be calculated in a straightforward way. In terms of the notation and classification introduced in proposition 3.2 we have

(i) $$\mu_0 = (1-q)(b-a)\frac{(q;q)_\infty\left(q\frac{b}{a};q\right)_\infty\left(q\frac{a}{b};q\right)_\infty\left(\frac{ab}{cd};q\right)_\infty}{\left(\frac{a}{c};q\right)_\infty\left(\frac{a}{d};q\right)_\infty\left(\frac{b}{c};q\right)_\infty\left(\frac{b}{d};q\right)_\infty} \tag{3.43}$$

(ii) $$\mu_0 = (1-q)(b-a)\frac{(q;q)_\infty\left(q\frac{b}{a};q\right)_\infty\left(q\frac{a}{b};q\right)_\infty}{\left(\frac{a}{c};q\right)_\infty\left(\frac{b}{c};q\right)_\infty} \tag{3.44}$$

(iii) $$\mu_0 = (1-q)(b-a)(q;q)_\infty\left(q\frac{b}{a};q\right)_\infty\left(q\frac{a}{b};q\right)_\infty \tag{3.45}$$

(iv) $$\mu_0 = (1-q)a^{r+1}\frac{(q;q)_r}{\left(\frac{a}{c};q\right)_{r+1}} \tag{3.46}$$

(v) $$\mu_0 = (1-q)a^{r+1}(q;q)_r \tag{3.47}$$

(vi)

(a) $$\mu_0 = (1-q)a^{r+1}\frac{(q;q)_\infty(-aq^{r+1};q)_\infty}{(-a;q)_\infty\left(-\frac{q}{a};q\right)_\infty} \tag{3.48}$$

(b) $$\mu_0 = (1-q)a^{r+1}\frac{(q;q)_\infty(aq^{r+1};q)_\infty}{(a;q)_\infty\left(\frac{q}{a};q\right)_\infty}. \tag{3.49}$$

Let us note that replacing r in (iv)–(vi) by $r+n \;\; n \in \mathbb{N}$, we obtain the moments μ_n for the corresponding cases.

In order to consider cases (i)–(iii) let us introduce a real function μ satisfying the equation

$$(1-\omega)\,B(Q)\,\mu(\omega) + (1-q)\,\omega Q A(Q)\,\mu(\omega) = 0. \tag{3.50}$$

It is easy to check that $\mu\left(q^{n+1}\right)$ satisfies the recurrence equation (3.41) and hence $\mu(q^{n+1}) = \mu_n$. Re-expressing (3.50) in the form

$$\left(\frac{B(Q)}{B\left(q^{-1}Q\right) - (1-q)\,q^{-1}QA\left(q^{-1}Q\right)} - \omega\right)\mu(\omega) = 0 \tag{3.51}$$

and the equation (3.7) in the form

$$\left(\frac{B(q\omega)}{B(\omega) - (1-q)\,\omega A(\omega)} - Q\right)\varrho(\omega) = 0 \tag{3.52}$$

one may observe some symmetry between the equation on ϱ and the equation for the moment function μ. After the substitution of the form $Q \to q\omega$ and $\omega \to Q$ the operator from (3.51) transforms into that of (3.52). Equation (3.51) as well as equation (3.52) can then easily be solved.

For example, if we assume that $B(1) = 0$ then (3.51) can be written in the form

$$\partial_{\mathcal{R}}\mu(\omega) = \mu(\omega) \tag{3.53}$$

where $\partial_{\mathcal{R}}$ is the $\mathcal{R}$-derivative. The function $\mathcal{R}$ is here given by

$$\mathcal{R}(\omega) = \frac{B(\omega)}{B(q^{-1}\omega) - (1-q)q^{-1}\omega A(q^{-1}\omega)}. \tag{3.54}$$

Then one of the two linearly independent solutions of (3.51) is simply the $\mathcal{R}$-exponential $\mathrm{Exp}_{\mathcal{R}}$. In this case it is given as the basic hypergeometric series

$$\mu_1(\omega) = \mathrm{Exp}_{\mathcal{R}}(\omega) = {}_3\Phi_2\left(\begin{smallmatrix}\frac{1}{a}, & 1, & q\\ \frac{1}{c}, & \frac{1}{d}\end{smallmatrix}; q; \omega\right) \tag{3.55}$$

where $a \neq 1$ is the root of the polynomial $B(\omega)$ and c and d are roots of the polynomial $B(q^{-1}\omega) - (1-q)q^{-1}\omega A(q^{-1}\omega)$. The function ${}_3\Phi_2$ is defined in [G-R, K-S]. The second solution $\mu_2(\omega)$ is related to $\mu_1(\omega)$ by the following formula (the Wronskian q-version):

$$\mu_2(\omega)\mu_1(q\omega) - \mu_2(q\omega)\mu_1(\omega) = x^{\lambda}\frac{(\alpha q; q)_\infty}{(\omega; q)_\infty} \tag{3.56}$$

where $q^{\lambda} = \frac{b_2}{b_0}$ and $\alpha = \frac{(1-q)a_1 - b_2}{b_0}$.

Any solution $\mu(\omega)$ is a linear combination of $\mu_1(\omega)$ and $\mu_2(\omega)$. We are then get the following formula for the moments:

$$\mu_n = \mu(q^{n+1}) = c_1\mu_1(q^{n+1}) + c_2\mu_2(q^{n+1}) \tag{3.57}$$

where the constants c_1 and c_2 are determined by

$$\mu_0 = \int_a^b \mathrm{d}\sigma_{AB} = c_1\mu_1(q) + c_2\mu_2(q) \tag{3.58}$$

$$a_0(c_1\mu_1(q) + c_2\mu_2(q)) + a_1(c_1\mu_1(q^2) + c_2\mu_2(q^2)) = 0. \tag{3.59}$$

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Appendix A. The affine difference calculus

In this section we present the preliminary considerations related to the calculus generated by the action of the affine group A_+ on the real line. Let us define the linear representation of A_+

$$(\mathfrak{L}_{q,h}\varphi)(x) := \varphi(qx+h) \qquad (q,h) \in A_+ = \{(q,h) : q > 0, h \in \mathbb{R}\} \tag{A.1}$$

acting on the functions φ from the $\mathfrak{F}$ algebra. Since our consideration will be formal in its character we do not impose any additional conditions on $\mathfrak{F}$.

According to [H] we introduce the derivative operator

$$(\partial_{q,h}\varphi)(x) := \frac{\varphi(x) - \varphi(qx+h)}{x - (qx+h)} \tag{A.2}$$

as a natural generalization of the q-derivative $\partial_q := \partial_{q,0}$ and of the difference derivative $\partial_h := \partial_{1,h}$.

The Leibnitz rule for the derivative $\partial_{q,h}$ is

$$\left(\partial_{q,h}\varphi\psi\right)(x) = \left(\partial_{q,h}\varphi\right)(x)\,\psi(x) + \left(\mathfrak{L}_{q,h}\varphi\right)(x)\left(\partial_{q,h}\psi\right)(x) \tag{A.3}$$

There is also the following equivariance property:

$$\mathfrak{L}_{c,t}^{-1}\circ\partial_{q,h}\circ\mathfrak{L}_{c,t} = c\,\partial_{q,ch+(1-q)t} \tag{A.4}$$

which enables us to reduce (q, h)-analysis to q-analysis. We have, for example

$$\partial_{q,h} = \mathfrak{L}^{-1}_{1,\frac{h}{1-q}}\circ\partial_q\circ\mathfrak{L}_{1,\frac{h}{1-q}}. \tag{A.5}$$

Let us now solve the equation

$$\partial_{q,h}\varphi = \varrho \tag{A.6}$$

for the given function $\varrho\in\mathfrak{F}$. In the order to do this, we apply the operator $\mathfrak{L}^k_{q,h}$ to (A.6) and find that

$$\mathfrak{L}^k_{q,h}\varphi(x) - \mathfrak{L}^{k+1}_{q,h}\varphi(x) = q^k\left[(1-q)x - h\right]\mathfrak{L}^k_{q,h}\varrho(x). \tag{A.7}$$

Summing up both sides of identity (A.7) with respect to k we get

$$\varphi(x) - \varphi\left(x^\infty\right) = \sum_{k=0}^{\infty}\left[x-(qx+h)\right]q^k\varrho\left(q^kx + \frac{1-q^k}{1-q}h\right) \tag{A.8}$$

where

$$x^\infty = \lim_{k\to\infty}\left(q^kx + \frac{1-q^k}{1-q}h\right) = \frac{h}{1-q}. \tag{A.9}$$

Equation (A.8) justifies the following definition of the (q, h)-integral:

$$\int_{q,h}\varrho(x) = \int_{x^\infty}^{x}\varrho(t)\,\mathrm{d}_{q,h}t := \sum_{k=0}^{\infty}\left[x-(qx+h)\right]q^k\varrho\left(q^kx+\frac{1-q^k}{1-q}h\right). \tag{A.10}$$

The (q, h)-integral operator is the right inverse of the (q, h)-derivative operator

$$\partial_{q,h}\circ\int_{q,h} = \mathrm{i\,d} \tag{A.11}$$

and, moreover,

$$\int_{q,h}\circ\partial_{q,h} = \mathrm{i\,d} - \delta_{1,\infty}. \tag{A.12}$$

The operator $\delta_{1,\infty}$ is an idempotent operator defined by

$$\left(\delta_{1,\infty}\varphi\right)(x) = \varphi\left(x^\infty\right) \tag{A.13}$$

projecting the function onto the constants.

As in (A.5) we have

$$\int_{q,h} = \mathfrak{L}^{-1}_{1,\frac{h}{1-q}}\circ\int_q\circ\mathfrak{L}_{1,\frac{h}{1-q}} \tag{A.14}$$

which reduces (by the translation authomorphism $\mathfrak{L}_{1,\frac{h}{q-1}}$) the (q, h)-integral to the Jackson integral $\int_q = \int_{q,0}$. The integration on the interval $[a, b]$ can be defined by

$$\int_a^b\varrho(t)\,\mathrm{d}_{q,h}t = \int_{b_\infty}^{b}\varrho(t)\,\mathrm{d}_{q,h}t - \int_{a_\infty}^{a}\varrho(t)\,\mathrm{d}_{q,h}t. \tag{A.15}$$

If $q\to 1$, the calculus presented above corresponds to the difference calculus. For $h\to 0$ one obtains a q-difference calculus. The differential calculus will be obtained when $q\to 1$, $h\to 0$.

Let us finally mention that the identities (A.5) and (A.14) enable us to reduce (q, h)-calculations to q-calculations. This property motivates us to discuss q-analysis.

Appendix B. Proof of theorem 3.2

$A \Leftrightarrow B$. The monic OPS $\{P_n(\omega)\}_{n=0}^{\infty}$ is uniquely defined by the weight function $\varrho(\omega)$ on the interval $[a, b]$. In order to prove the equivalence of the properties A and B it is sufficient to show that the system of polynomials defined by (3.18) is a monic OPS. In order to do that let us reexpress the function

$$F_k(\omega; n) := \partial_q^k\big[\varrho(\omega)\, B(\omega) B\big(q^{-1}\omega\big) \ldots B\big(q^{-(n-1)}\omega\big)\big] \tag{B.1}$$

in the following way:

$$F_k(\omega; n) = \varrho(\omega) B(\omega)\, B\big(q^{-1}\omega\big) \ldots B\big(q^{-(n-1-k)}\omega\big) R_{k,n}(\omega) \quad k = 0, 1, \ldots, n-1 \tag{B.2}$$

$$F_n(\omega; n) = \varrho(\omega) R_{n,n}(\omega) \tag{B.3}$$

where $R_{k,n}(\omega)$ is a polynomial of degree not greater than $k \leqslant n$. These polynomials satisfy the recurrence formula

$$\begin{aligned} R_{k+1,n}(\omega) = {} & A(\omega) R_{k,n}(q\omega) + B\big(q^{-(n-1-k)}\omega\big)\partial_q R_{k,n}(\omega) \\ & + \frac{B\left(q^{-(n-1-k)}\omega\right) - B(\omega)}{(1-q)\,\omega} R_{k,n}(q\omega) \end{aligned} \tag{B.4}$$

for $k = 0, 1, \ldots, n-1$ with the initial condition $R_{0,n}(\omega) \equiv 1$.

For $k < n$, applying (B.2) we have

$$\begin{aligned} \int_a^b q^{\frac{k(k+1)}{2}} \omega^k \tilde{P}_n(\omega) \varrho(\omega)\, \mathrm{d}_q\omega &= c_n \int_a^b q^{\frac{k(k+1)}{2}} \omega^k \partial_q^n F_0(\omega; n)\, \mathrm{d}_q\omega \\ &= c_n q^{\frac{k(k+1)}{2}} \omega^k F_{k-1}(\omega; n)\,|_a^b - c_n [k] \int_a^b q^{\frac{k(k+1)}{2}} \omega^{k-1} \partial_q^{n-1} F_0(\omega; n)\, \mathrm{d}_q\omega \\ &= c_n q^{\frac{k(k+1)}{2}} \omega^k \varrho(\omega) B(\omega) B\big(q^{-1}\omega\big) \ldots B\big(q^{-(n-1-(k-1))}\omega\big) R_{k-1,n}(\omega)|_a^b \\ &\quad - c_n [k] \int_a^b q^{\frac{k(k+1)}{2}} \omega^{k-1} \partial_q^{n-1} F_0(\omega; n)\, \mathrm{d}_q\omega \\ &= \cdots = (-1)^k [k][k-1] \ldots [1]\, c_n \int_a^b \partial_q^{n-k} F_0(\omega; n)\, \mathrm{d}_q\omega \\ &= (-1)^k [1] \ldots [k]\, \varrho(\omega) B(\omega) B\left(q^{-1}\omega\right) \ldots B\big(q^{-(n-1-(k-1))}\omega\big)|_a^b = 0. \end{aligned} \tag{B.5}$$

This shows that the polynomials $\tilde{P}_n(\omega)$, $n \in \mathbb{N} \cup \{0\}$ form an OPS. With the correct normalizing constants c_n one can obtain the monic OPS.

$B \Rightarrow C$. We have proved the validity of the q-Rodrigues formula for any Pearson data and thus by theorem 3.1 we have

$$\frac{1}{[n]}\partial_q \tilde{P}_n(\omega) = c_{n-1}^{(1)} \frac{1}{\varrho^{(1)}(\omega)} \partial_q^{n-1}\big[\varrho^{(1)}(\omega) B^{(1)}(\omega) B^{(1)}\big(q^{-1}\omega\big) \ldots B^{(1)}\big(q^{-(n-2)}\omega\big)\big] \tag{B.6}$$

for the Pearson data $\big(A^{(1)}(\omega), B^{(1)}(\omega)\big)$ given by (3.9) and (3.10). Using now the equality

$$Q\varrho(\omega) B(\omega) \ldots B\big(q^{-(n-1)}\omega\big) = \varrho^{(1)}(\omega) B^{(1)}(\omega)\, B^{(1)}\big(q^{-1}\omega\big) \ldots B^{(1)}\big(q^{-(n-2)}\omega\big) \tag{B.7}$$

and substituting (B.6) into (3.18) we find

$$\begin{aligned} \tilde{P}_n(\omega) &= c_n \frac{1}{\varrho(\omega)} \partial_q^n\big[Q^{-1}\varrho^{(1)}(\omega) B^{(1)}(\omega)\, B^{(1)}\big(q^{-1}\omega\big) \ldots B^{(1)}\big(q^{-(n-2)}\omega\big)\big] \\ &= c_n q^{-(n-1)} \frac{1}{\varrho(\omega)} \partial_q Q^{-1} \partial_q^{n-1}\big[\varrho^{(1)}(\omega) B^{(1)}(\omega)\, B^{(1)}\big(q^{-1}\omega\big) \ldots B^{(1)}\big(q^{-(n-2)}\omega\big)\big] \end{aligned}$$

$$= c_n q^{-(n-1)} \frac{1}{\varrho(\omega)} \partial_q Q^{-1} \frac{[n]\,\varrho^{(1)}(\omega)}{c^{(1)}_{n-1}} \partial_q \tilde{P}_n(\omega)$$
$$= \frac{c_n}{c^{(1)}_{n-1}} q^{-(n-1)} [n] \frac{1}{\varrho(\omega)} \partial_q \varrho(\omega) B(\omega) Q^{-1} \partial_q \tilde{P}_n(\omega)$$
$$= \frac{c_n}{c^{(1)}_{n-1}} q^{-(n-1)} [n] \left[A(\omega)\partial_q \tilde{P}_n(\omega) + B(\omega)\partial_q Q^{-1} \partial_q \tilde{P}_n(\omega)\right]. \tag{B.8}$$

We have proved (3.19). In order to show (3.20) we compare the coefficients of x^n on both sides of (3.19). Additionally we have the formula

$$\frac{c^{(1)}_{n-1}}{c_n} = [n]\, q^{-(n-1)} \lambda_n \tag{B.9}$$

for the normalizing coefficients. Later we will use (B.9) for the calculation of c_n.

$C \Rightarrow B$. The proof is one of induction. It is easy to see that for $n = 1$ (3.18) follows from (3.19). Let us assume that it is true for $n-1$. We now prove it for n.

From our assumption and theorem 3.1 we have

$$\frac{1}{[n]} \partial_q \tilde{P}_n(\omega) = c^{(1)}_{n-1} \frac{1}{\varrho^{(1)}(\omega)} \partial_q^{n-1} \left(\varrho^{(1)}(\omega) B(\omega) \ldots B\left(q^{-(n-2)}\omega\right)\right). \tag{B.10}$$

Using (3.18) and proposition 3.1, we obtain a thesis after simple calculations.

$C \Leftrightarrow D$. From (3.20) we have

$$\tilde{P}_n(\omega) = \frac{[n]}{\lambda_n} \left(A(\omega) + B(\omega)\partial_q Q^{-1}\right) \frac{1}{[n]} \partial_q \tilde{P}_n(\omega). \tag{B.11}$$

According to theorem 3.1, the polynomials $\left\{\frac{1}{[n]}\partial_q \tilde{P}_n(\omega)\right\}_{n=0}^{\infty}$ form the monic OP with respect to $\left(A^{(1)}(\omega), B^{(1)}(\omega)\right)$ given by (3.9) and (3.10). We can thus apply formula (B.11) with $\left(A^{(1)}(\omega), B^{(1)}(\omega)\right)$ to the polynomial $\frac{1}{[n]}\partial_q P_n(\omega)$. Repeating this procedure n-times and using the formula

$$\lambda^{(k)}_n = a^{(k)}_1 [n] + b_2 [n] [n-1] q^{-(n-1)} \tag{B.12}$$

where $a^{(k)}_1$ is defined by (3.22) we obtain (3.21).

$B \Rightarrow E$. In order to prove that the recurrence formula (3.23) holds we use the identity

$$\partial_q^k \left[\varrho(\omega) B(\omega)\, B\left(q^{-1}\omega\right) \ldots B\left(q^{-(n-1)}\omega\right)\right]$$
$$= \varrho(\omega) B(\omega)\, B\left(q^{-1}\omega\right) \ldots B\left(q^{-(n-1-k)}\omega\right) R_{k,n}(\omega) \tag{B.13}$$

where the polynomial $R_{k,n}(\omega)$ satisfies the recurrence equation (B.4). From equation (B.13) and the Rodrigues formula one has

$$\tilde{P}_n(\omega) = c_n R_{n,n}(\omega)\,. \tag{B.14}$$

Let us denote by α_k, β_k and γ_k the three highest coefficients of the polynomial

$$R_{k,n}(\omega) = \alpha_k \omega^k + \beta_k \omega^{k-1} + \gamma_k \omega^{k-2} + \cdots. \tag{B.15}$$

After substituting (B.15) into (B.14) and comparing the coefficients of the monomials ω^{k+1}, ω^k and ω^{k-1} we obtain the following system of recurrence equations:

$$\begin{aligned}
\alpha_{k+1} &= (a_1 - b_2 [-2n + 2 + k])\, q^k \alpha_k \\
\beta_{k+1} &= (a_0 - b_1 [-n + 1])\, q^k \alpha_k + (a_1 - b_2 [-2n + 3 + k])\, q^{k-1} \beta_k \\
\gamma_{k+1} &= b_0 [k]\, \alpha_k + (a_0 - b_1 [-n + 2])\, q^{k-1} \beta_k + (a_1 - b_2 [-2n + 4 + k])\, q^{k-2} \gamma_k
\end{aligned} \tag{B.16}$$

which can be solved by iteration as follows:

$$\alpha_k = q^{\frac{k(k-1)}{2}} \prod_{l=0}^{k-1} (a_1 - b_2\,[-2n+2+l]) \tag{B.17}$$

$$\beta_k = \frac{[n]}{q^{n-1}} \cdot \frac{a_0 - b_1\,[-n+1]}{a_1 - b_2\,[-2n+2]} \cdot \alpha_k \tag{B.18}$$

$$\gamma_k = \frac{(1-q^n)\left(1-q^{n-1}\right)}{(1-q)^2\,(1+q)} \times \frac{(a_0 - b_1\,[-n+2])\,(a_0 - b_1\,[-n+1]) + b_0\,(a_1 - b_2\,[-2n+2])}{(a_1 - b_2\,[-2n+2])\,(a_1 - b_2\,[-2n+3])}\alpha_k. \tag{B.19}$$

Thus for the monic polynomial

$$\tilde{P}_n(\omega) = \omega^n + \beta\left(q^n\right)\omega^{n-1} + \gamma\left(q^n\right)\omega^{n-2} + \cdots \tag{B.20}$$

we find that the coefficients

$$\beta\left(q^n\right) := \frac{\beta_n}{\alpha_n}$$
$$\gamma\left(q^n\right) := \frac{\gamma_n}{\alpha_n}$$

are given by the rational functions (3.3) and (3.4). Using the three-term recurrence relation (3.24) and (B.15) we obtain the formulae (3.1) and (3.2) for the structural functions $\mathfrak{R}$ and $\mathfrak{D}$.

$E \Rightarrow A$. The recurrence formula (3.23) rewriten for the orthonormal polynomials $\{P_n(\omega)\}_{n=0}^{\infty}$ takes the form (2.28) which means that the Hamiltonian H_{red} (2.22) has a Jacobi matrix of type D. Thus (see [A]) this Hamiltonian is essentially selfadjoint and has a simple spectrum. This shows that there is a unique measure $\mathrm{d}\sigma_{AB}$ such that

$$\int_a^b P_n(\omega) P_m(\omega)\,\mathrm{d}\sigma_{AB}(\omega) = \delta_{nm}. \tag{B.21}$$

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